

Recall $\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$, for $\operatorname{Re}(s) > 1$.

We've already seen that $\zeta(s)$ admits analytic continuation to $\operatorname{Re}(s) > 0$. Today we prove meromorphic continuation to $s \in \mathbb{C}$ and functional equation.

Theorem: The function $\zeta(s)$ admits meromorphic continuation to the entire complex plane and verifies the functional equation

$$\xi(s) := \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) = \xi(1-s).$$

The only pole of $\zeta(s)$ is at $s=1$, it is simple with the residue $\operatorname{res}_{s=1} \zeta(s) = 1$.

Motivation for proof:

We note that for $\operatorname{Re}(s) > 1$, we have

$$\begin{aligned} \xi(s) &= \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \sum_{n \geq 1} (\pi n^2)^{-s/2} \Gamma\left(\frac{s}{2}\right) \\ &= \sum_{n \geq 1} (\pi n^2)^{-s/2} \mathcal{M}(e^{-y})\left(\frac{s}{2}\right) \\ &= \sum_{n \geq 1} \mathcal{M}(e^{-\pi n^2 y})\left(\frac{s}{2}\right) \end{aligned}$$

$$= \mathcal{M} \left(\underbrace{\sum_{n \geq 1} e^{-\pi n^2 y}}_{w(y)} \right) \left(\frac{s}{2} \right).$$

$$= \mathcal{M}(w(y)) \left(\frac{s}{2} \right).$$

Define $\theta: (0, \infty) \rightarrow \mathbb{R}$ given by

$$\theta(x) := \sum_{n \in \mathbb{Z}} e^{-\pi n^2 x}.$$

It converges absolutely for $x > 0$:

$$|\theta(x)| \leq 1 + \int_{-\infty}^{\infty} e^{-\pi y^2 x} dy \leq 1 + \frac{1}{\sqrt{x}}.$$

Also, for $x \geq 1$:

$$\begin{aligned} \sum_{n=1}^{\infty} e^{-\pi n^2 x} &= e^{-\pi x} \sum_{n \geq 1} e^{-\pi(n^2-1)x} \leq e^{-\pi x} \sum_{n \geq 1} e^{-\pi(n^2-1)} \\ &\ll e^{-\pi x}. \end{aligned}$$

Therefore $\theta(x) = 1 + O(e^{-\pi x})$, for $x \geq 1$.

Hence $w(x) = \frac{\theta(x) - 1}{2} \ll e^{-\pi x}$, for $x \geq 1$.

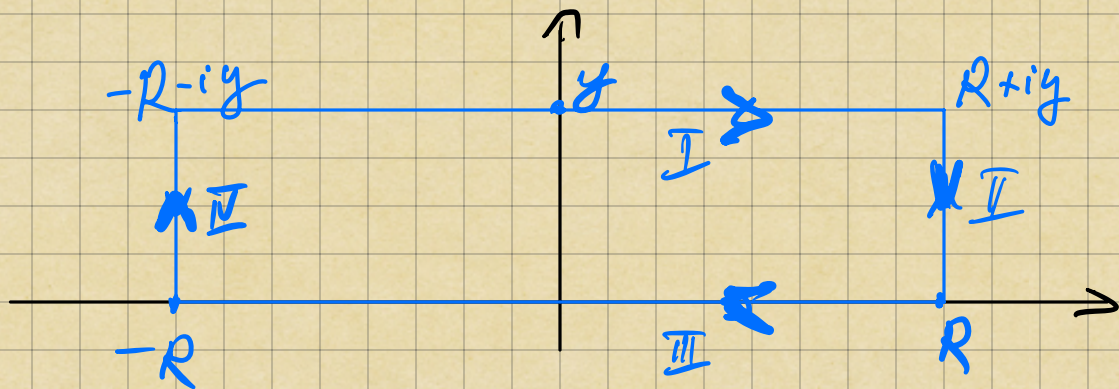
Lemma: Let $f(x) = e^{-\pi x^2} \in \mathcal{S}(\mathbb{R})$.

Then $\hat{f}(y) = e^{-\pi y^2} = f(y)$.

Proof: By completing the square,

$$\hat{f}(y) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x y} = e^{-\pi y^2} \int_{-\infty}^{\infty} e^{-\pi(x+iy)^2} dx$$

$f(z) = e^{-\pi z^2}$ is a holomorphic function for $z \in \mathbb{C}$



From Cauchy's theorem:

$$\int_{-R+iy}^{R+iy} f(z) dz + \int_{R+iy}^R f(z) dz + \int_R^{-R} f(z) dz + \int_{-R}^{-R+iy} f(z) dz = 0.$$

Note that $|f(z)| = e^{-\pi(\operatorname{Re}(z)^2 - \operatorname{Im}(z)^2)}$.

Hence the second and fourth integral tend to 0 as $R \rightarrow \infty$.

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-\pi(x+iy)^2} dx &= \lim_{R \rightarrow \infty} \int_{-R+iy}^{R+iy} f(z) dz = - \lim_{R \rightarrow \infty} \int_R^{-R} f(z) dz \\ &= \int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1. \quad \square \end{aligned}$$

Theorem (Functional equation for $\theta(x)$)

For all $x > 0$, we have $\theta(x) = \frac{1}{\sqrt{x}} \theta\left(\frac{1}{x}\right)$.

Proof: We've just seen $f(x) = e^{-\pi x^2} \in S(\mathbb{R})$
and $\hat{f}(x) = f(x)$.

Therefore, for $c > 0$, let

$$f_c(x) = e^{-\pi c x^2} = f(\sqrt{c}x) \in S(\mathbb{R})$$

$$\text{Hence } \hat{f}_c(y) = \frac{1}{\sqrt{c}} \hat{f}\left(\frac{y}{\sqrt{c}}\right) = \frac{1}{\sqrt{c}} e^{-\frac{\pi y^2}{c}}.$$

From Poisson summation, we see

$$\begin{aligned} \theta(x) &= \sum_{n \in \mathbb{Z}} e^{-\pi n^2 x} = \sum_{n \in \mathbb{Z}} f_x(n) = \sum_{n \in \mathbb{Z}} \hat{f}_x(n) \\ &= \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{x}} e^{-\frac{\pi n^2}{x}} = \frac{1}{\sqrt{x}} \theta\left(\frac{1}{x}\right). \quad \square \end{aligned}$$

Proof of main theorem:

$$\text{Denote } w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x} = \frac{\theta(x) - 1}{2}.$$

$$\text{Then we have } w\left(\frac{1}{x}\right) = \sqrt{x} w(x) + \frac{\sqrt{x} - 1}{2}.$$

(From functional equation of $\theta(x)$).

Hence, for $\operatorname{Re}(s) > 1$, we have

$$\begin{aligned}
\Xi(s) &= \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \mathcal{M}(w(x))\left(-\frac{s}{2}\right) \\
&= \int_0^{\infty} w(x) x^{s/2} \frac{dx}{x} \\
&= \int_1^{\infty} w(x) x^{s/2} \frac{dx}{x} + \int_1^{\infty} w\left(\frac{1}{x}\right) x^{-\frac{s}{2}} \frac{dx}{x} \\
&= \int_1^{\infty} w(x) x^{s/2} \frac{dx}{x} + \int_1^{\infty} \left(\sqrt{x} w(x) + \frac{\sqrt{x}-1}{2}\right) x^{-\frac{s}{2}} \frac{dx}{x} \\
&= -\frac{1}{s} + \frac{1}{s-1} + \int_1^{\infty} w(x) \left(x^{\frac{s}{2}} + x^{\frac{s-1}{2}}\right) \frac{dx}{x} \quad (*)
\end{aligned}$$

We note that the integral is absolutely and uniformly convergent for all $s \in \mathbb{C}$, (since $w(x) \ll e^{-\pi x}$), hence it defines a holomorphic function.

In (*) we can interchange $s \leftrightarrow 1-s$, so the functional equation follows.

The only poles of $\Xi(s)$ are at 0 and 1, and they are simple.

There are no zeros of $\Xi(s)$ in $\Re(s) > 1$

(Since $\Gamma(s)$ has no zeros and $\zeta(s)$ has no zeros in $\Re(s) > 1$)

Therefore by functional equation, $\xi(s)$ has no zeros in $\operatorname{Re}(s) < 0$.

Corollary (Zeros and poles of $\zeta(s)$)

$\zeta(s)$ is a meromorphic function with

- A simple pole at $s=1$ and no other poles.
- 'Trivial zeros' at $s=-2, -4, -6, \dots$ and no other zeros in $\operatorname{Re}(s) < 0$.
- 'Non-trivial' zeros ρ with $\operatorname{Re}(\rho) \in [0, 1]$.
- No zeros with $\operatorname{Re}(s) > 1$.

Proof: We have that
$$\zeta(s) = \frac{\pi^{s/2} \Gamma\left(\frac{1-s}{2}\right)}{\pi^{(1-s)/2} \Gamma\left(\frac{s}{2}\right)} \zeta(1-s).$$

$\pi^{s/2}$ and $\pi^{(1-s)/2}$ have no poles or zeros in complex plane
 $\Gamma\left(\frac{s}{2}\right)$ has simple poles at $s=0, -2, -4, \dots$ and no zeros.

Conclusion follows.

□

The location of the zeros of a complex variable gives us a lot of information. We'll see later some important consequences for $\zeta(s)$ and the distribution of primes. We begin by reviewing some facts from complex analysis.

Lemma (Jensen's inequality)
(Size of analytic function controls density of zeros)

Let $f(z)$ an analytic function on the disk $|z| \leq R$. If $f(0) \neq 0$, the number of zeros of f in the disk $|z| < R/2$ is bounded by

$$2 \max_{|z|=R} \log \left| \frac{f(z)}{f(0)} \right|.$$

Proof: Let $z_1, \dots, z_k$ denote the zeros of f in the disk $|z| < R/2$.

Note that if $|z|=R$, then

$$|z - z_j| = |\bar{z} - \bar{z}_j| = \left| \frac{z\bar{z} - z\bar{z}_j}{z} \right| = \left| \frac{R^2 - z\bar{z}_j}{R} \right|.$$

$$\text{Let } h(z) = f(z) \prod_{j=1}^k \left(\frac{R^2 - z\bar{z}_j}{R(z - z_j)} \right).$$

Then $h(z)$ analytic on $|z| \leq R$ and $|h(z)| = |f(z)|$
if $|z| = R$.

By maximum modulus principle,

$$\max_{|z|=R} |f(z)| = \max_{|z|=R} |h(z)| \geq |h(0)| = |f(0)| \prod_{j=1}^k \frac{R}{|z_j|} \geq |f(0)| 2^k.$$

$$\text{Therefore } k \leq \frac{1}{\log 2} \max_{|z|=R} \log \left| \frac{f(z)}{f(0)} \right|. \quad \square$$

We denote $N(T)$ the number of non-trivial zeros of $\zeta(s)$ with $|Im(s)| \leq T$.

Lemma (Zeros of $\zeta(s)$ are not too dense)

For $T \geq 2$, we have $N(T) - N(T-1) \ll \log T$
and $N(T) \ll T \log T$.

Proof: We know that for $Re(s) > 0$,

$$\zeta(s) = \frac{1}{s-1} + 1 - s \int_1^{\infty} \frac{\zeta(y)}{y^{s+2}} dy.$$

This implies that for $\sigma \geq \frac{1}{2}$ and $|t| \geq 1$,
we have $|\zeta(\sigma+it)| \ll |t|$.

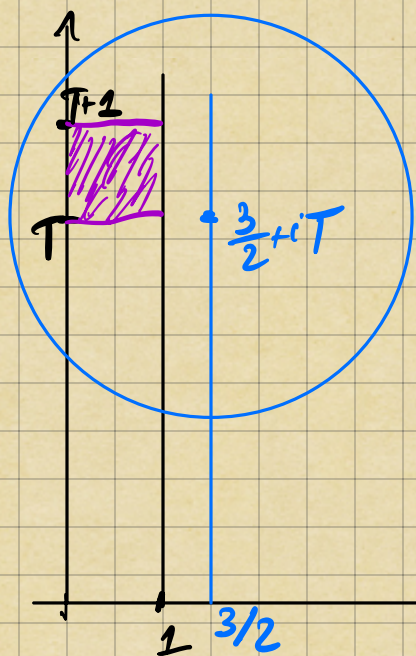
We can integrate by parts, and obtain

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} - s(s+1) \int_1^{\infty} \frac{\zeta(y)^2 - \zeta(y)}{2y^{s+2}} dy.$$

This is well defined for $\operatorname{Re}(s) > -1$
 and it implies that if $\sigma \geq -\frac{1}{2}$, $|t| \geq 1$,
 $|\zeta(\sigma+it)| \ll (1+|t|)^2$.

We can iterate the process of integration by parts, and for any $N \in \mathbb{N}$, we have
 that if $\sigma > -N + \frac{1}{2}$, $|t| \geq 1$
 $|\zeta(\sigma+it)| \ll_N (1+|t|)^{N+1}$.

Suppose $T \geq 10$ (conclusion follows otherwise
 since there are finitely many non-trivial zeros
 with $|\operatorname{Im} s| \leq 10$).



Let $g(z) = \zeta(\frac{3}{2} + z + iT)$. Then
 $g(z)$ is analytic in $|z| \leq 4$ and it
 satisfies $|g(z)| \ll T^4$ in this disk.

Also $|g(0)| = |\zeta(\frac{3}{2} + iT)| \neq 0$.

and $\frac{1}{|g(0)|} = \left| \frac{1}{\zeta(\frac{3}{2} + iT)} \right|$

$$= \left| \zeta\left(\frac{3}{2} + iT\right) \right| = \left| \sum_{n \geq 1} \frac{\mu(n)}{n^{3/2 + iT}} \right| \leq \sum_{n \geq 1} \frac{1}{n^{3/2}} = \zeta\left(\frac{3}{2}\right).$$

From Jensen's inequality, $g(z)$ can have $O(\log T)$ zeros with $|z| \leq 2$. This implies $g(s)$ has at most $O(\log T)$ zeros with $T \leq \Re s \leq T+1$.

(Since $\{0 \leq \Re(s) \leq 1, T \leq \Im(s) \leq T+1\} \subset \{s: |s - \frac{3}{2} - iT| \leq 2\}$).

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